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A piecewise deterministic process for homodyne measurement of an atom

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Abstract. The piecewise deterministic process associated with event enhanced quantum theory is constructed in the case of homodyne detection of a two-level atom and the counting distribution is obtained. The non-Markovian property of the photocounting statistics is proved and the sub-Poissonian character of event registration established.

1. Introduction

A significant technological advance, which has been made in experimental physics, allows for continuous observation of individual quantum systems (QSs) [1–5]. In most such experiments, results are obtained in the form of a distribution of measurement events (usually registration of photons: e.g., [1, 2, 4]). It is also the case that the distributions are derived from prolonged observation of individual QSs [1, 2]. Consequently, we cannot use the standard von Neuman measurement theory, but we have to employ a theory which enables analysis of the behaviour of individual QSs. Following a general theory of open systems [6, 7], the event-enhanced quantum theory (EEQT) was proposed and developed [8–12]. EEQT describes the measurement of a QS as an evolution of a total system generated by a completely positive semigroup. The total system is a coupled quantum–classical system, where the classical part represents a measurement device and the quantum one an investigated system. They are described by an Abelian and non-Abelian algebra of observables, respectively.

In order to provide the reconstruction of measurement events, a piecewise deterministic (PD) process has been established [10]. In contrast to other approaches to the event generation algorithm (e.g., as a diffusion process [20] or by numerical methods [21]) it has been shown [10] that this process is a unique PD process which, after averaging, reproduces the evolution of the coupled quantum–classical system. The uniqueness and explicit presence of events in the process are essential for deducing the PD algorithm describing real experiments.

In this paper statistical properties of the homodyne measurement of a two-level atom are investigated by means of a PD process associated to EEQT. We show that the rigorously derived stochastic process of event registration is not a Markovian one. Despite this, it reconstructs the same probability of sample history as in [28]. We find that the photocounting statistics is sub-Poissonian not only in the case of a driven atom [3, 4, 24, 26] but also during the homodyne detection of a state of an atom. We show that this feature does not depend on the time of detection. We find that not every state can be observed by the typical set of homodyne detection. However, if outcomes of photoelements are treated properly the density matrix of the atom can be calculated. A similar result might be obtained by optical homodyne tomography (OHT) [22, 23, 25]; nevertheless in our case we need only two count distributions, whereas in OHT a set of such distributions is required.

2. The formalism of EEQT

We start with a brief description of the EEQT and the PD process associated with EEQT. Detailed descriptions of EEQT and the PD process may be found in [9] and [10], respectively. The basic idea is to couple two systems (quantum and classical) and describe their evolution in terms of a completely positive semigroup.

The classical system

We will consider the simplest case of a discrete set of classical events. The elements of the set of pure states are labelled by indices $\alpha \in X_{Cl}$, where X_{Cl} is a discrete and countable space. Statistical states of the system are probability measures on X_{Cl}: i.e., sequences $\{p_\alpha\}$ which satisfy

$$\forall_{\alpha\in X_{Cl}}\, p_\alpha \geqslant 0 \qquad \text{and} \qquad \sum_{\alpha\in X_{Cl}} p_\alpha = 1.$$

The algebra of observables of a classical system (CS) is a C^*-algebra $\mathcal{A}_{Cl}$ of complex functions on X_{Cl}: here $\mathcal{A}_{Cl}$ is isomorphic to l^∞, the algebra of all uniformly bounded sequences. Since we want to use the Hilbert space language, even for the description of the CS, we construct the Hilbert space of the CS by choosing the set of base vectors labelled by the elements of X_{Cl} and taking the closure of the set of all linear combinations of the base vectors.

The quantum system

The observable algebra of a QS is $\mathcal{A}_q = L(\mathcal{H}_q)$—the algebra of bounded operators on a Hilbert space $\mathcal{H}_q$. Pure states of QS are unit vectors in $\mathcal{H}_q$. We assume that proportional vectors describe the same quantum state. They form a complex projective space $CP(\mathcal{H}_q)$ over $\mathcal{H}_q$. Statistical states of QS are positive operators $\hat{\rho}$ on $\mathcal{H}_q$ with $\mathrm{Tr}(\hat{\rho}) = 1$.

The total system

We take the tensor product of the algebras of observables for the algebra of observables of the total system: $\mathcal{A}_{\mathrm{tot}} = \mathcal{A}_q \otimes \mathcal{A}_{Cl}$. It acts on the tensor product $\mathcal{H}_q \otimes \mathcal{H}_{Cl}$. In our case $\mathcal{A}_{\mathrm{tot}}$ can be thought of as an algebra of diagonal matrices $A = (a_{\alpha\beta})$, whose entries are operators $a_{\alpha\alpha} \in \mathcal{A}_q$ and $a_{\alpha\beta} = 0$ for $\alpha \neq \beta$. Statistical states of the total system are diagonal matrices $\rho = \mathrm{diag}\,(\rho_0, \ldots, \rho_n, \ldots)$, whose entries are positive operators on $\mathcal{H}_q$. The statistical states satisfy the normalization $\mathrm{Tr}\,(\rho) = \sum_\alpha \mathrm{Tr}\,(\rho_\alpha) = 1$. Duality between observables and states is given by the expectation value $\langle A\rangle_\rho = \sum_\alpha \mathrm{Tr}\,(A_\alpha \rho_\alpha)$.

The dynamics of the total system

We have to distinguish two situations: when no information is transferred from the QS to the CS and when such a transfer takes place. In the first situation, quantum dynamics is described by the Hamiltonian $H_\alpha : \mathcal{H}_\alpha \to \mathcal{H}_\alpha$ (it may depend on the state of the CS). Using matrix notation we write $H = \mathrm{diag}\,(H_\alpha)$.

Since the CS has been chosen as discrete it cannot have a continuous-time dynamics of its own†. Coupling of the system is specified by a matrix $V = (g_{\alpha\beta})$, where $g_{\alpha\beta}$ are linear operators: $g_{\alpha\beta} : \mathcal{H}_\alpha \to \mathcal{H}_\beta$. We assume the evolution of the system is given by completely positive (CP) semigroup α^t, $t \geqslant 0$ of CP maps of the algebra of observables with the property $\alpha^t(I) = I$. In view of theorems by Stinspring and Lindblad [14, 15] any norm-continuous semigroup of CP which maps on $\mathcal{A}_{\mathrm{tot}}$ is of the form: $\alpha^t = \exp(tL)$, where

$$L(A) = \mathrm{i}[H_\alpha, A_\alpha] + \sum_\beta g^*_{\beta\alpha} A_\beta g_{\beta\alpha} - \tfrac{1}{2}\{\Lambda_\alpha, A_\alpha\} \tag{1}$$

or, equivalently,

$$\dot{\rho} = -\mathrm{i}[H_\alpha, \rho_\alpha] + \sum_\beta g_{\alpha\beta}\rho_\beta g^*_{\alpha\beta} - \tfrac{1}{2}\{\Lambda_\alpha, \rho_\alpha\} \tag{2}$$

where

$$\Lambda_\alpha = \sum_\beta g^*_{\beta\alpha} g_{\beta\alpha}. \tag{3}$$

Brackets [,],{,} denote the commutator and anticommutator, respectively.

The PD process associated with EEQT

The Liouville equation (2) can be obtained as an expectation value of a PD process. A detailed mathematical description of this process may be found in [10, 16]. The evolution between jumps is governed by a complete vector field X on the total system. We need two more components to define the process: jump rate and a transition kernel Q. The vector field X generates a flow $\Phi(t, x)$ in E (the pure state space of the total system), which is given by $\Phi(t, x) = \gamma_x(t)$, where $\gamma_x(t)$ is the integral curve of X starting at the point $x \in E$. The jump rate is a measurable function $\lambda : E \to R_+ \cup \{0\}$ such that for any $x \in E$ the mapping $t \to \lambda \circ \phi(t, x)$ is integrable at least near $t = 0$. The set of those $x \in E$ for which $\lambda(x) = 0$ we denote by E_0. The transition kernel $Q : \mathcal{B}(E) \times E \to [0, 1]$ satisfies the following conditions:

(1) $Q(E, x) = 1, \forall x \in E$.
(2) $Q(\{x\}, x) = 0$ if $x \in E \setminus E_0$ and $Q(\{x\}, x) = 1$ for $x \in E_0$.
(3) $\forall \Gamma \in \mathcal{B}(E)$ the map is measurable.

Here $\mathcal{B}(E)$ denotes the Borel σ-algebra on E. In our case‡ $E = \dot{\cup} CP_\alpha$, $\alpha = 0, 1, \ldots, n$ and

$$Xf(\Psi, \alpha) = \frac{\mathrm{d}}{\mathrm{d}t} f\left(\frac{\exp(-\mathrm{i}H_\alpha t - \frac{1}{2}\Lambda_\alpha t)\Psi}{||\exp(-\mathrm{i}H_\alpha t - \frac{1}{2}\Lambda_\alpha t)\Psi||}, \alpha\right)\Bigg|_{t=0} \tag{4}$$

$$\lambda(\Psi, \alpha) = \langle \Psi, \Lambda_\alpha \Psi \rangle \tag{5}$$

$$Q(\mathrm{d}\phi, \beta; \Psi, \alpha) = \frac{||g_{\alpha\beta}\Psi||^2}{\lambda(\Psi, \alpha)} \delta\left(\phi - \frac{g_{\alpha\beta}\Psi}{||g_{\alpha\beta}\Psi||}\right) \mathrm{d}\phi. \tag{6}$$

The triple (X, λ, Q) is called the local characteristic of the process.

Using the above characteristics of the process we can obtain physically interesting characteristics of the process [16]. Let us define two sequences of measurable random variables:

$$T_n : \Omega_x \to [0, \infty] \qquad T_n(\omega) = t_n \qquad X_n(\omega) = x_n$$

† It is possible to consider the case when the set of classical events is a continuous one, e.g. [13].
‡ CP_α—a complex projective space $CP(\mathcal{H}_q)$ over $\mathcal{H}_q$.

where $\Omega_x = \{\omega \in \Omega; x_0 = x\}$, Ω is the set of all sequences $\omega = (t_0, x_0; t_1, x_1; \ldots)$ and $t_0 = 0$, $t_0 \leqslant t_1 \leqslant t_2 \leqslant \ldots, t_n \in [0, \infty]$, $x_n \in E$, $n \in \mathcal{N} \cup \{0\}$. We can interpret T_n as a time when n jumps took place.

By standard methods of stochastic processes (for the rigorous derivation, see [16]) we derive the following conditional expectations:

$$E_x[1_{\{X_1 \subset \Gamma\}} | T_1] = Q(\Gamma, \phi(T_1, x)) \tag{7}$$

$$E_x[1_{\{X_{n+1} \subset \Gamma\}} | X_n, T_{n+1}] = Q(\Gamma, \phi(T_{n+1}, X_n)) \tag{8}$$

$$E_x[1_{\{T_{n+1} \leqslant 1\}} | T_n, X_n] = \begin{cases} 0 & \text{if} \quad t < T_n \\ 1 - \exp(-\Lambda(t - T_n, X_n)) & \text{if} \quad t \geqslant T_n \end{cases} \tag{9}$$

and the probability of events:

$$P_x[T_1 \leqslant t] = 1 - \exp(-\Lambda(t, x)) \tag{10}$$

$$P_x[T_{n+1} - T_n > s] = E_x[\exp(-\Lambda(s, X_n))] \tag{11}$$

where $\Lambda(t, x) := \int_0^t \lambda(\phi(s, x))\,\mathrm{d}s$. So we are able to calculate any physically interesting distributions by applying (7–11): e.g., the probability of counting only one event up to time t reads

$$P_x[T_2 > t \wedge T_1 \leqslant t] = \int_0^t \int_E \lambda(\phi(u, x)) \exp(-\Lambda(u, x)) \times \exp(-\Lambda(t - u, y)) Q(\mathrm{d}y, \phi(u, x))\,\mathrm{d}u.$$

3. The homodyne measurement

We construct a stochastic process describing measurement of the state of an atom by means of homodyne detection. Homodyne measurement is a powerful tool of quantum optics used for observation of very weak quantum beams [17] (usually under the sensitivity of photoelements) or special properties of QS, e.g. squeezed states [26]. The basic idea is to disrupt the beam with a local oscillator by means of a beam splitter [27].

For the sake of simplicity we restrict ourselves to the case where we neglect the influence of driving field, cavity etc. We consider only the mutual interaction of the atom and the homodyne measurement†.

The EEQT description of the homodyne measurement

The CS consists of two photodetectors, whereas a two-level atom constitutes the QS. The state space of the QS is CP^1 and that of the CS is represented by $\mathcal{N}^2$ (we assume that $\{0\} \in \mathcal{N}$), so the pure state space of the coupled quantum–classical system is given by $\bigcup_{n,m=0}^{\infty} CP^1$.

We define the evolution of the system by the following assumptions:

$$H = 0$$

$$g_{(i,j)(k,l)} = \begin{cases} \sqrt{(\frac{\gamma}{2})}(A + \mathrm{i}\beta) & \text{when } \mathrm{i} = k + 1,\ j = l \\ \sqrt{(\frac{\gamma}{2})}(A - \mathrm{i}\beta) & \text{when } \mathrm{i} = k,\ j = l + 1 \\ 0 & \text{in other cases} \end{cases} \tag{12}$$

where H is the Hamiltonian of the QS; $g_{(i,j)(k,l)}$ are the coupling operators $A = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$; γ is the coupling constant of the QS and CS; and β is the coupling constant of the QS and local oscillator. The above assumptions mean that we neglect the possibility of simultaneous

† Outlines of methods of homodyne detection may be found in various works, e.g. [28].

measurement of a photon by both detectors. The evolution of the QS is given by a modified Schrödinger equation:

$$\dot{\Psi}_t = (-\mathrm{i}H - \tfrac{1}{2}\Lambda)\Psi_t \tag{13}$$

where H is given by (12) and $\Lambda = \gamma(A^\dagger A + |\beta|^2)$. A solution for Ψ_t can be written as $\Psi_t = \hat{U}(t)\Psi_0$, where

$$\hat{U}(t) = \mathrm{e}^{-\frac{1}{2}\Lambda t} = \mathrm{e}^{-\frac{1}{2}\gamma(A^\dagger A + |\beta|^2)t}. \tag{14}$$

Equation (14) can be written in the explicit form [28] as

$$\hat{U}(t) = \begin{pmatrix} \exp(-\frac{\gamma}{2}t) & 0 \\ 0 & 1 \end{pmatrix}. \tag{15}$$

We define the excited and ground states as $\Psi_e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\Psi_g = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Using the above assumptions and the general procedure we can set the PD process.

4. The stochastic process

First, we will investigate the stochastic properties of the system and compare the homodyne measurement with the detection of coherent light.

Using the assumptions of the system from section 3 and a general procedure [10] we define the ingredients of the PD process in the following way. The jump rate for all (i, j) is

$$\lambda(\Psi, (i, j)) = \langle \Psi, \Lambda\Psi \rangle. \tag{16}$$

The transition kernel is

$$\begin{aligned} Q(\mathrm{d}\phi, (i, j); \Psi, (k, l)) = {} & \frac{\delta_i^{k+1}\delta_j^l}{2} \frac{||(A + \mathrm{i}\beta)\Psi||^2}{||A\Psi||^2 + |\beta|^2} \delta\left(\phi - \frac{(A + \mathrm{i}\beta)\Psi}{||(A + \mathrm{i}\beta)\Psi||}\right) \mathrm{d}\phi \\ & + \frac{\delta_i^k \delta_j^{l+1}}{2} \frac{||(A - \mathrm{i}\beta)\Psi||^2}{||A\Psi||^2 + |\beta|^2} \delta\left(\phi - \frac{(A - \mathrm{i}\beta)\Psi}{||(A + \mathrm{i}\beta)\Psi||}\right) \mathrm{d}\phi. \end{aligned} \tag{17}$$

The deterministic flow is given by

$$\Phi(t, (\Psi, (i, j))) = \frac{\hat{U}(t)\Psi}{||\hat{U}(t)\Psi||^2}. \tag{18}$$

We begin investigation of the system by finding the probability of observing one possible history of measurement: i.e., to register a sequence $(j_1, \ldots, j_N)$, where $j_i = \pm 1$. If $j_i = 1$, then the ith registration was done by detector DI (when $j_i = -1$ by DII). We denote the probability of observing a sequence $(j_1, \ldots, j_N)$ within the time interval $(0, t)$ by $p^t(j_1, \ldots, j_N)$. Then

$$\begin{aligned} p^t(j_1, \ldots, j_N) = {} & \int_0^t \mathrm{d}t_N \int_0^{t_N} \mathrm{d}t_{N-1} \ldots \int_0^{t_2} \mathrm{d}t_1 \int_E \ldots \int_E \mathrm{e}^{-\Lambda(t_1, \alpha_0)} \lambda(\phi(t_1, \alpha_0)) \\ & \times Q(j_1, \phi(t_1, \alpha_0)) \mathrm{e}^{-\Lambda(t_2 - t_1, \alpha_1)} \lambda(\phi(t_2 - t_1, \alpha_1)) \\ & \times Q(j_2, \phi(t_2 - t_1, \alpha_1)) \times \cdots \times \mathrm{e}^{-\Lambda(t_{N-1} - t_{N-2}, \alpha_{N-2})} \\ & \times \lambda(\phi(t_{N-1} - t_{N-2}, \alpha_{N-2})) Q(j_{N-2}, \phi(t_{N-1} - t_{N-2}, \alpha_{N-2})) \\ & \times \mathrm{e}^{-\Lambda(t_N - t_{N-1}, \alpha_{N-1})} \lambda(\phi(t_N - t_{N-1}, \alpha_{N-1})) \\ & \times Q(j_{N-1}, \phi(t_N - t_{N-1}, \alpha_{N-1})) \mathrm{e}^{-\Lambda(t - t_N, \alpha_N)}. \end{aligned} \tag{19}$$

In order to simplify (19) we use (16), (17) and the following expression:

$$\exp(-\Lambda(t, (\Psi_\alpha, \alpha))) = \exp\left(-\int_0^t \mathrm{d}s\, \phi(s, (\Psi_\alpha, \alpha) \right)$$
$$= \exp\left(-\int_0^t \left\langle \frac{\hat{U}(s)\Psi_\alpha}{||\hat{U}(s)\Psi_\alpha||}, \gamma(A^\dagger A + |\beta|^2)\frac{\hat{U}(s)\Psi_\alpha}{||\hat{U}(s)\Psi_\alpha||} \right\rangle \right). \tag{20}$$

Because

$$\frac{\mathrm{d}}{\mathrm{d}s}||\hat{U}(s)\Psi||^2 = \langle \hat{U}(s)\Psi, -\gamma(A^\dagger A + |\beta|^2)\hat{U}(s)\Psi \rangle \tag{21}$$

we get

$$\exp(-\Lambda(t, (\Psi_\alpha, \alpha)) = \exp\left(\int_0^t \frac{\frac{\mathrm{d}}{\mathrm{d}s}||\hat{U}(s)\Psi||^2}{||\hat{U}(s)\Psi||^2}\,\mathrm{d}s \right) = ||\hat{U}(t)\Psi||^2 \tag{22}$$

and obtain (19) in the form

$$p^t(j_1, \ldots, j_N) = \int_0^t \mathrm{d}t_N \int_0^{t_N} \mathrm{d}t_{N-1} \ldots \int_0^{t_2} \mathrm{d}t_1 ||\hat{U}(t - t_N)(A + j_N \mathrm{i}\beta)$$
$$\times \hat{U}(t_N - t_{N-1}) \ldots \hat{U}(t_2 - t_1)(A + j_1 \mathrm{i}\beta)\hat{U}(t_1)\Psi_0||^2. \tag{23}$$

If we use the explicit form of Ψ_0

$$\Psi_0 = \begin{pmatrix} a \\ b \end{pmatrix} \tag{24}$$

with the condition $||\Psi_0|| = 1$ we obtain the equation

$$p^t(j_1, \ldots, j_N) = \left(\frac{\gamma}{2}\right)^N \frac{|\beta|^{2N} t^N}{N!}(|b|^2 + |a|^2 \mathrm{e}^{-\gamma t})\mathrm{e}^{-\gamma|\beta|^2 t}$$
$$+ \left(\frac{\gamma}{2}\right)^N |\beta|^{2(N-1)}|a|^2 \mathrm{e}^{-\gamma|\beta|^2 t} \sum_{k=1}^N \int_0^t \mathrm{d}s_N \int_0^{s_N} \mathrm{d}s_{N-1} \ldots \int_0^{s_2} \mathrm{d}s_1\, \mathrm{e}^{-\gamma s_k}$$
$$+2\left(\frac{\gamma}{2}\right)^N |\beta|^{2(N-1)}|a|^2 \mathrm{e}^{-\gamma|\beta|^2 t} \sum_{k<l} j_k j_l \int_0^t \mathrm{d}s_N \int_0^{s_N} \mathrm{d}s_{N-1} \ldots \int_0^{s_2} \mathrm{d}s_1\, \mathrm{e}^{-\gamma \frac{s_k + s_l}{2}}$$
$$+\mathrm{i}\left(\frac{\gamma}{2}\right)^N |\beta|^{2(N-1)}(\beta a^* b - \beta^* a b^*)\mathrm{e}^{-\gamma|\beta|^2 t}$$
$$\times \sum_{l=1}^N j_l \int_0^t \mathrm{d}s_N \int_0^{s_N} \mathrm{d}s_{N-1} \ldots \int_0^{s_2} \mathrm{d}s_1\, \mathrm{e}^{-\gamma s_l}. \tag{25}$$

Properties of the stochastic process

Using expression (19) we can construct many random variables and calculate their expectation values. We investigate the simplest case of the sum and difference of detector counts. The first one is important because it allows us to compare photodetection of coherent light and the system used for homodyne measurement. In the second, we treat the outputs of photodetectors in the same manner as in the case of the homodyne measurement.

We define the counting measurement as the result of summing the number of detections made by detectors DI and DII.

The contrasting measurement is the difference between the counting done by detectors DI and DII. This is exactly what is obtained in homodyne measurement where we observe the difference of photocurrents between both photoelements.

The counting measurement

We find the probability of obtaining N counts in time t by

$$p_N^H(t) = \sum_{j_1,\ldots,j_N=\pm 1} p^t(j_1,\ldots,j_N). \tag{26}$$

Substituting (25) into (26) we get

$$\begin{aligned} p_N^H(t) = \gamma^N \frac{|\beta|^{2N} t^N}{N!} (|b|^2 + |a|^2 \mathrm{e}^{-\gamma t}) \mathrm{e}^{-\gamma|\beta|^2 t} \\ + |a|^2 \frac{\gamma^{N-1} |\beta|^{2(N-1)} t^{N-1}}{(N-1)!} (1 - \mathrm{e}^{-\gamma t}) \mathrm{e}^{-\gamma|\beta|^2 t}. \end{aligned} \tag{27}$$

Equation (27) can be rewritten as

$$p_N^H(t) = \frac{(\gamma|\beta|^2 t)^N}{N!} \mathrm{e}^{-\gamma|\beta|^2 t} (|b|^2 + |a|^2 \mathrm{e}^{-\gamma t}) + \frac{(\gamma|\beta|^{2)} t)^{N-1}}{(N-1)!} \mathrm{e}^{-\gamma|\beta|^2 t} (1 - |b|^2 - |a|^2 \mathrm{e}^{-\gamma t}). \tag{28}$$

It is easy to see that the stochastic process (28) can be defined as a superposition of two independent Markovian processes. One is the Poissonian process: $p_N^P(t) = \frac{(\gamma|\beta|^2 t)^N}{N!} \mathrm{e}^{-\gamma|\beta|^2 t}$; and the second is the irreversible (birth–death) process with the probability of registration of a state: $q_1 = 1 - q_0$ where $q_0 = |b|^2 + |a|^2 \mathrm{e}^{-\gamma t}$. This process can be thought of as a registration process with efficiency $1 - |b|^2$. The transition matrix for this process is given by

$$p_{00}(s,t) = \frac{q_0(t)}{q_0(s)} \tag{29}$$

$$p_{10}(s,t) = \frac{q_0(s) - q_0(t)}{q_0(s)} \tag{30}$$

$$p_{01}(s,t) = 0 \tag{31}$$

$$p_{11}(s,t) = 1. \tag{32}$$

Using the above description it is straightforward to compare the process associated with homodyne measurement and the Poissonian one. We use the quantity defined in [5]:

$$Q = \frac{\overline{\Delta n^2}}{\overline{n}} - 1. \tag{33}$$

Because the Poisson process and the irreversible one are independent we have

$$X = X_1 + X_2 \tag{34}$$

$$\overline{X} = \overline{X_1} + \overline{X_2} \tag{35}$$

$$\overline{\Delta X^2} = \overline{\Delta X_1^2} + \overline{\Delta X_2^2} \tag{36}$$

where X, X_1, X_2 denote the PD process, the Poisson process and the birth–death process, respectively. Therefore,

$$\overline{X_1} = \gamma|\beta|^2 t \tag{37}$$

$$\overline{X_2} = |a|^2 (1 - \mathrm{e}^{-\gamma t}) \tag{38}$$

$$\overline{\Delta X_1^2} = \gamma|\beta|^2 t \tag{39}$$

$$\overline{\Delta X_2^2} = |a|^2 (1 - \mathrm{e}^{-\gamma t})(1 - |a|^2 (1 - \mathrm{e}^{-\gamma t})). \tag{40}$$

So the quantity Q_H for the homodyne measurement process reads

$$Q_H = \frac{-|a|^4 (1 - \mathrm{e}^{-\gamma t})^2}{\gamma|\beta|^2 t + |a|^2 (1 - \mathrm{e}^{-\gamma t})} \tag{41}$$

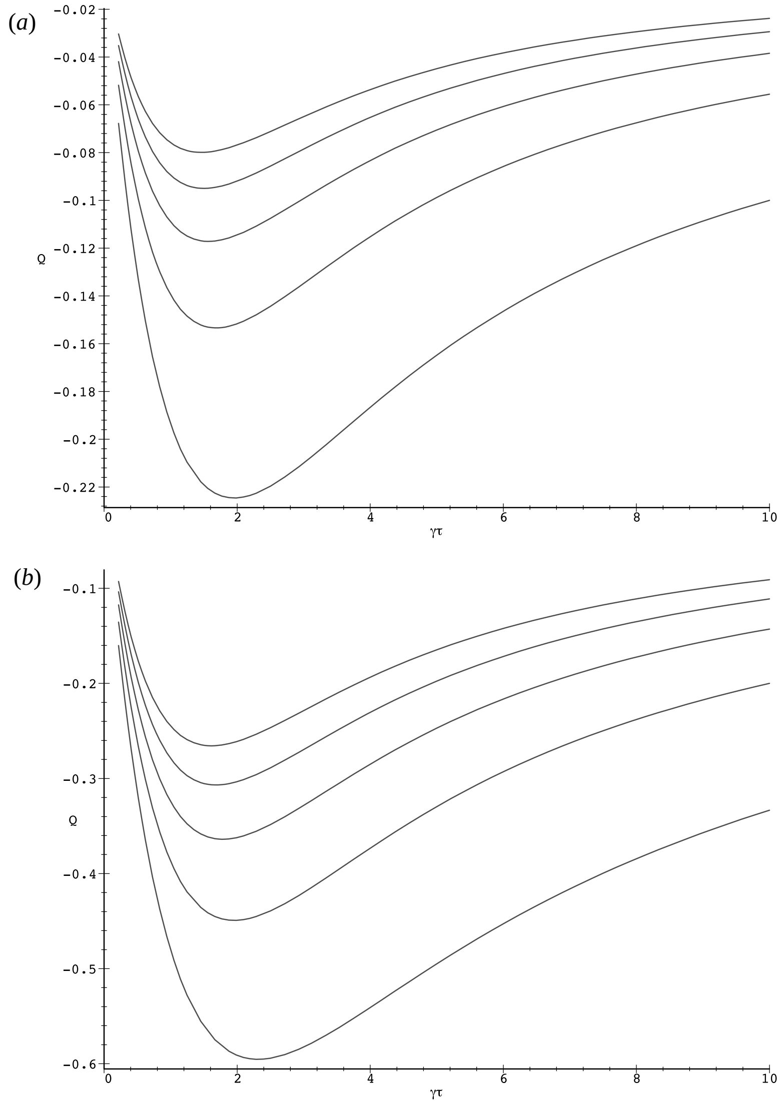

Figure 1. Comparison of Q_H plotted for different values of $|\beta|^2$ (from 0.2 to 1 in steps of 0.2 beginning with the lowest). (*a*) $|a|^2 = 0.5$ and (*b*) $|a|^2 = 1$.

and implies that $p_N^H(t)$ is a sub-Poissonian process.

From figure 1 we see that the observability of the sub-Poissonian statistics of the distribution strongly depends on the value of the coupling constant $|\beta|^2$. However, comparing the experimentally obtained data [5] and figure 1 we see that sub-Poissonian statistics may be easily observed, even in the case of a state different from an excited one and with a high value of $|\beta|^2$, although the best results will be obtained in the case of a properly chosen time of observation. We present the plots of Q_H for different initial values of $|a|^2$ and the coupling constant $|\beta|^2$.

Using short counting time, equation (41) may be written as

$$Q_H = -\gamma t \frac{|a|^4}{|\beta|^2 t + |a|^2}. \tag{42}$$

This coincides exactly with the result obtained in [30] for photodetection, but in the case of long counting time we obtain the convergence to the Poissonian process because $\lim_{t\to\infty} Q_H = 0$, whereas for photodetection [30] $Q_D \neq 0$ even for a long time.

Stochastic properties of the counting measurement

The transition probability function of the process (28) is given by

$$\begin{aligned} p((n, t_0), (n+k, t_0+t)) = {} & \frac{p_n^P(t_0) q_0(t_0)}{p_n^H(t_0)} (p_k^P(t) q_0(t|t_0) + p_{k-1}^P(t) q_1(t|t_0)) \\ & + \frac{p_{n-1}^P(t_0) q_1(t_0)}{p_n^H(t_0)} p_k^P(t) \end{aligned} \tag{43}$$

for all $k \geqslant 0$ and zero for $k < 0$. $p((n, t_0), (n+k, t_0+t))$ is the probability of registering $n+k$ events during the interval $(0, t_0+t)$ under the condition of observing n events up to time t_0; $q_0(t|t_0)$ is the conditional probability that in the time t there is no registration (under the condition that up to time t_0 there was no counting) and $q_1(t|t_0)$ is the conditional probability of a registration particle within period (t_0, t) (under the same condition as above). We define these quantities as follows:

$$q_0(t|t_0) = \frac{q_0(t_0+t)}{q_0(t_0)} \tag{44}$$

$$q_1(t|t_0) = \frac{q_1(t_0+t) - q_1(t_0)}{q_0(t_0)}. \tag{45}$$

The transition probability (43) satisfies

$$\sum_{k=0}^{\infty} p((n, t_0), (n+k, t_0+t)) = 1. \tag{46}$$

Proof of (46) is straightforward by substituting equation (43), using properties of the Poissonian process and definition of the conditional probability. The second property is

$$\sum_{k=0}^{\infty} p((n, t_0), (l, t_0+t)) p_n^H(t_0) = p_l^H(t_0+t). \tag{47}$$

In order to prove (47) it is enough to substitute (43) into (47) and use the property of the Poissonian process: $\sum_{n=0}^{\infty} p_n^P(t_0) p_{l-n}^P(t) = p_l^P(t_0+t)$. However, it is worth noting that the transition probability function (43) is not a Markovian one. This can be easily checked by showing that equation (43) does not satisfy the following Chapman–Kolmogorov equation:

$$p((n_3, t_3), (n_1, t_1)) = \sum_{n_2=0}^{\infty} p((n_3, t_3), (n_2, t_2)) p((n_2, t_2), (n_1, t_1)). \tag{48}$$

However, as was pointed out before, the PD process associated to EEQT is a Markov process.

From the physical point of view it is interesting to derive two other characteristics of the process: a transition function $p(n, t, t+T)$ and the expectation value of counts.

The transition function $p(n, t, t+T)$, gives us the probability of measuring n events in the time interval $(t, t+T)$:

$$p(n, t, t+T) = \sum_{k=0}^{\infty} p((k, t), (n+k, t+T)) p_k^H(t). \tag{49}$$

Substituting (43) into (49) and using properties of the Poissonian process we get

$$p(n,t,t+T) = p^P_{n-1}(T) + (p^P_n(T) - p^P_{n-1}(T))(q_0(t+T) + q_1(t)) \tag{50}$$

or, in equivalent form,

$$p(n,t,t+T) = p^P_n(T)(q_0(t+T) + q_1(t)) + p^P_{n-1}(T)(1 - q_0(t+T) - q_1(t)). \tag{51}$$

From (51) we see that the probability of counting n events within the time interval $(t, t+T)$ is the sum of two possible events: one when all counting is done by the Poissonian process and the second when $n-1$ are done by the Poissonian process and one by the irreversible counting process.

The expectation value of counts for the probability distribution (26) is given by

$$p^H_+(t) = \gamma|\beta|^2 t(|b|^2 + |a|^2 \mathrm{e}^{-\gamma t}) + |a|^2(1 - \mathrm{e}^{-\gamma t})(\gamma|\beta|^2 t + 1). \tag{52}$$

In the case of Ψ_e and Ψ_g equation (52) simplifies to

$$\Psi_e \Rightarrow p^H_+(t) = 1 + \gamma|\beta|^2 t - \mathrm{e}^{-\gamma t} \tag{53}$$

$$\Psi_g \Rightarrow p^H_+(t) = \gamma|\beta|^2 t. \tag{54}$$

The expectation value of the contrasting measurement

We obtain the probability that the difference between the counting of detectors DI and DII is R under the condition that the total number of counted states is N as

$$p^H_{R,N}(t) = \sum_{j_1+\cdots+j_N=R} p^t(j_1,\ldots,j_N). \tag{55}$$

In order to simplify calculation of the expectation value we introduce the following quantity:

$$v^H_{R,N}(t) = \sum_{j_1+\cdots+j_N=R} p^t(j_1,\ldots,j_N) - \sum_{j_1+\cdots+j_N=-R} p^t(j_1,\ldots,j_N). \tag{56}$$

Substituting (25) into (56) and performing all necessary calculation we get

$$v^H_{R,N}(t) = 2\mathrm{i}\left(\frac{\gamma}{2}\right)^N |\beta|^{2(N-1)}(\beta a^* b - \beta^* a b^*)\mathrm{e}^{-\gamma|\beta|^2 t}$$
$$\times \sum_{j_1+\cdots+j_N=R}\sum_{l=1}^{N} j_l \int_0^t \mathrm{d}s_N \int_0^{s_N} \mathrm{d}s_{N-1} \ldots \int_0^{s_2} \mathrm{d}s_1\, \mathrm{e}^{-\gamma\frac{s_l}{2}}. \tag{57}$$

When we sum over the index j_l with the condition $j_1 + \cdots + j_N = R$ the number of indices j_l with +1 and −1 is, respectively,

$$k = \frac{N+R}{2} \tag{58}$$

$$l = \frac{N-R}{2}. \tag{59}$$

Since there are $w(+)$ and $w(-)$ possible combinations with the signs + and −, where

$$w(+) = \frac{N!}{(k-1)!l!} \tag{60}$$

and

$$w(-) = \frac{N!}{k!(l-1)!} \tag{61}$$

so

$$\sum_{j_1+\cdots+j_N=R} j_k = w(+) - w(-) = \binom{N}{k}(2k - N). \tag{62}$$

Using (62) we calculate that

$$\sum_{j_1+\cdots+j_N=R} \sum_{l=1}^{N} j_l \int_0^t \mathrm{d}s_N \int_0^{s_N} \mathrm{d}s_{N-1} \ldots \int_0^{s_2} \mathrm{d}s_1 \, \mathrm{e}^{-\gamma \frac{s_l}{2}}$$
$$= \sum_{l=1}^{N} \binom{N}{k} (2k-N) \int_0^t \mathrm{d}s_N \int_0^{s_N} \mathrm{d}s_{N-1} \ldots \int_0^{s_2} \mathrm{d}s_1 \, \mathrm{e}^{-\gamma \frac{s_l}{2}}$$
$$= \binom{N}{k} (2k-N) \frac{t^{N-1}}{\gamma(N-1)!} (1-\mathrm{e}^{-\gamma t}). \tag{63}$$

By substitution of (63) into (56)

$$v_{R,N}^{H}(t) = 2\mathrm{i} \left(\frac{\gamma}{2}\right)^N |\beta|^{2(N-1)} (\beta a^* b - \beta^* a b^*) \mathrm{e}^{-\gamma|\beta|^2 t}$$
$$\times \binom{N}{k} (2k-N) \frac{t^{N-1}}{\gamma(N-1)!} (1-\mathrm{e}^{-\gamma t}) \tag{64}$$

and we obtain eventually the expectation value of this measurement:

$$p_{-}^{H}(t) = -\mathrm{i}(\beta a^* b - \beta^* a b^*)(\gamma|\beta|^2 t + 1)(1-\mathrm{e}^{-\gamma t}). \tag{65}$$

In the case of the excited and ground states:

$$\Psi_e \Rightarrow p_{-}^{H}(t) = 0 \tag{66}$$

and

$$\Psi_0 \Rightarrow p_{-}^{H}(t) = 0. \tag{67}$$

5. Observability of states

One of the most interesting features of the investigated measurement is the possibility of receiving two kinds of information.

We see, by the comparison of equation (52) and (65), that in the case of the counting measurement we receive information about the absolute value of a component of the initial state of the atom. On the other hand, the contrasting measurement, usually called the homodyne measurement, allows us to measure the initial state of the system. Unfortunately, we cannot observe all of the states in this way. The measurable states must obey the condition

$$\beta a^* b - \beta^* a b^* \neq 0. \tag{68}$$

Equation (65) allows us to optimize the system (the coupling constants β, α) to adjust the device to an expected initial state. However, this is not the only information which could be gained from the results of both kinds of experiments. Equation (68) can be expanded in the form

$$\mathrm{i}(\beta a^* b - \beta^* a b^*) = 2(\beta_R a_I b_R - \beta_R a_R b_I - \beta_I a_R b_R - \beta_I a_I b_I) \tag{69}$$

where $\beta_R, \beta_I, a_R, a_I, b_R, b_I$ are real and imaginary components of β, a, b, respectively. If we perform two measurements on the same QS by two different values of coupling constant β we obtain

$$\beta_I = 0 \Rightarrow \mathrm{i}(\beta a^* b - \beta^* a b^*) = 2\beta_R(a_I b_R - a_R b_I) \tag{70}$$

and

$$\beta_R = 0 \Rightarrow \mathrm{i}(\beta a^* b - \beta^* a b^*) = -2\beta_I(a_R b_R + a_I b_I). \tag{71}$$

So these two measurements allow us to fully characterize the density matrix of the measured system. This is a significant simplification of the optical homodyne tomography [25], which required a set of count distributions.

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